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UGC CARE II

CONVERGENCE THEOREM AND OBTAINED EXPANSION FOR THE VECTOR FUNCTION OF SECOND ORDER DIFFERENTIAL EQUATIONS

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ABSTRACT

The first quarter of Nineteenth Century is considered as a basis to the theory of boundary value problems, which involves the determination of solutions of differential equations that satisfy prescribed boundary conditions. By the application of the method of separation of variable to partial differential equations of mathematical physics, one was led to the expansion of an arbitrary function in terms of a system of functions known as 'proper functions' or 'eigen functions' of a differential equation for corresponding 'proper values' or 'eigne values' of an involved parameter. In this paper contains a convergence theorem and obtained expansion for the vector function of the type $f\ (x) = \begin{bmatrix} f_1(x) \\ f_2(x) \end{bmatrix}$ *which is continuous in some suitable interval and bounded variation in that interval, when p(x) and q(x) tends to + ∞ or -∞, which will be suitable in the work.*

Keywords: convergence theorem, differential equations, eigen functions expansions etc.

INTRODUCTION

The work of mathematical physicist J.B.J. Fourier (1758-1830) and mathematical astronomer, F.W. Bessel (1784-1846) in the first quarter of Nineteenth Century is considered as a basis to the theory of boundary value problems, which involves the determination of solutions of differential equations that satisfy prescribed boundary conditions. By the application of the method of separation of variable to partial differential equations of mathematical physics, one was led to the expansion of an arbitrary function in terms of a system of functions known as 'proper functions' or 'eigen functions' of a differential equation for corresponding 'proper values' or 'eigne values' of an involved parameter.

The theory of eigen function expansions associated with the second order differential equations goes far back to the time of Sture and Liouville, *i.e.*, more than a century ago. The modern theory of singular differential operator was first developed by N.Neyl (1885-1955) on singular self-and joint Linear differential operator of the second order and later on developed by M.H. Stone, J. Von Newmann (1905-1957), K. Friedrichs, K.Kodaira.

Work on boundary value problems associated with self-adjoint differential system due to David Hilbert (1862-1941) was fundamental one. But the discussion on the simultaneous system was started by either in the early 20th century. Schlesinger [53] took a system of a linear differential equations of the first order with coefficient to which one rational in X and obtained the asymptotic forms for a solution. Harwitz [38] considered the simultaneous expansion of two functions in terms of solutions of a pair of differential equations of the first order with restricted boundary conditions. Mirkhoff and Langer [9] and Bliss [10] considered the possibilities of simultaneously expanding n arbitrary functions in terms of the solutions of a property restricted type of first order differential equations with a number of boundary conditions at one or both ends of a finite interval.

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TWO LINEARLY INDEPENDENT SOLUTIONS

The two linearly independent solutions of the system

$$(M+\lambda)\emptyset = 0 \ \text{(where, } 0 \le X < \infty) \ldots\ldots\ldots\ldots(1)$$

which are L^2 [0,∞], are given by

$$\psi_r(x,\lambda) = \theta_r(x,\lambda) + \sum_{s=1}^{2} m_{rs}(\lambda)\emptyset_s(x,\lambda), \quad (r = 1,2)\ldots\ldots(2)$$

Now by u_j(x, λ), v_j(x, λ) (j=1,2) are large the imaginary part of λ is large and positive and

$$\left.\begin{aligned} M_{j1}(\lambda) &\sim -\frac{u_j(10)}{2}, if\ u_j(0) \neq 0, \\ &\sim -\frac{u_j(10)}{2i\mu}, if\ u_j(0) = 0, \\ M_{j2}(\lambda) &\sim -\frac{v_j(0)}{2}, if\ v_j(0) \neq 0, \\ &\sim -\frac{v_j(0)}{2i\mu}, if\ v_j(00) = 0 \end{aligned}\right\}\ldots\ldots\ldots(3)$$

So $\emptyset_j$(x,λ) (j=1,2) are not L^2 [0,∞). But from

$$X_j(x) = e^{i\mu x}\{C_{j1}(\lambda) + 0(1)\}, \quad (j = 1,2)\ldots\ldots(4)$$

and $T_j(x) = e^{i\mu y}\{C_{j2}(\lambda) + 0(1)\}, \quad (j = 1,2)\ldots\ldots\ldots(5)$

we see that Xj (x,λ), Yj (x,λ) (j=1,2) are small when imaginary part of λ is large and positive. There we conclude that are linearly independent. Then

$$\psi_r(x,\lambda) = \sum_{s=1}^{2} K_{rs}(\lambda)\beta_s(x,\lambda) + \sum_{s=1}^{2} \ell_{rs}(\lambda)\phi_s(x,\lambda), (r = 1,2) \quad (6)$$

Since B_r(x,λ) (r=1,2) are L^2 [0,∞), but $\emptyset_j$(x, λ) (j=1,2) are not L^2 [0,∞), therefore $l_{rs}(\lambda) = 0(1 \le r, B \le 2)$.

Hence,

$$\psi_r(x,\lambda) = \sum_{s=1}^{2} K_{rs}(\lambda)\beta_s(x,\lambda), \qquad (r = 1,2) \quad (7)$$

From the asymptotic formula (4) and (5) we obtain, as x → ∞

$$u_j'(x,\lambda) \sim -\,i\mu\, e^{-i\mu x_{M_{j1}}(\lambda)}$$

$$v_j'(x,\lambda) \sim -\,i\mu\, e^{-i\mu x_{M_{j2}}(\lambda)}$$

$$X_j'(x,\lambda) \sim -\,i\mu\, e^{-i\mu x_{C_{j1}}(\lambda)}$$

$$Y_j'(x,\lambda) \sim -\,i\mu\, e^{-i\mu x_{C_{j2}}(\lambda)} \quad (8)$$

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Where dashes denote differentiation with respect to x. Using (4), (5), (7) and (8), we obtain from

$$\left[\emptyset_j\,(0|x_2\lambda)\psi_r(x,\lambda)\right] = \delta_{jr^2}(1 \le j, r \ \le 2)\text{..........(9)}$$

We get,

$$K_{11}(M_{11}C_{11} + M_{12}C_{12}) + K_{12}(M_{11}C_{21} + M_{12}C_{11}) + \frac{1}{2i\mu} = 0$$

$$\left.\begin{aligned} K_{11}(M_{11}C_{11} + M_{12}C_{12}) + K_{12}(M_{11}C_{21} + M_{12}C_{11}) + \tfrac{1}{2i\mu} = 0 \\ K_{11}(M_{21}C_{11} + M_{22}C_{12}) + K_{12}(M_{21}C_{21} + M_{22}C_{22}) = 0 \\ K_{21}(M_{11}C_{11} + M_{12}C_{12}) + K_{12}(M_{11}C_{21} + M_{12}C_{11}) = 0 \\ K_{21}(M_{21}C_{11} + M_{22}C_{12}) + K_{22}(M_{21}C_{21} + M_{22}C_{22}) + \tfrac{1}{2i\mu} = 0 \end{aligned}\right\}\text{..............(10)}$$

From (10) we get equation (11) as,

$$K_{11} = \frac{(M_{21}C_{21} + M_{22}C_{22})}{2i\mu(M_{11}M_{22} - M_{12}M_{21})(C_{12}C_{21} - C_{11}C_{22})}$$

$$K_{12} = \frac{(M_{21}C_{11} + M_{12}C_{22})}{2i\mu(M_{11}M_{22} - M_{12}M_{21})(C_{12}C_{21} - C_{11}C_{22})}$$

$$K_{21} = \frac{(M_{11}C_{21} + M_{12}C_{22})}{2i\mu(M_{11}M_{22} - M_{12}M_{21})(C_{12}C_{21} - C_{11}C_{22})}$$

$$K_{22} = \frac{(M_{11}C_{11} + M_{12}C_{12})}{2i\mu(M_{11}M_{22} - M_{12}M_{21})(C_{12}C_{21} - C_{11}C_{22})}$$

CONVERGENCE THEOREM

If $f(x) = \begin{pmatrix} f_1(x) \\ f_2(x) \end{pmatrix}$ be a real-valued vector of bounded variation in 0≤x< ∞ and be L^2 [0, ∞) and L^2 [0, ∞) and L^2 [0, ∞) and λ ≠ and eigenvalue of the system (1), then

$$\frac{1}{2}\,\{f(x+0) + f(x-0)\} = -\frac{1}{\pi i}\lim_{R\to\infty}\int_{-R+i\epsilon}^{R+i\epsilon}\Phi(x,\lambda), d\lambda \qquad (12)$$

Uniformly for 0 < ∈≤ 1, where

$$\emptyset(x,\lambda) = \begin{pmatrix} \emptyset_1(x,\lambda) \\ \emptyset_2(x,\lambda) \end{pmatrix} = \int_0^\infty G(x,y;\lambda) f(y)dy \qquad (13)$$

We prove convergence theorem for ϕ_1(x, λ) because similar result holds for ϕ_2 (x, λ). Now we write ϕ_1(x, λ) as

$$\Phi_2(x,\lambda) = \psi_{11}(x,\lambda)\int_0^x \phi_1^T f(y)\,dy + \psi_{21}(x,\lambda)\int_0^x \phi_1^T f(y)\,dy +$$

$$+u_1(x,\lambda)\int_x^\infty \psi_1^T(y,\lambda)\, f(y)dy + u_2(x,\lambda)\int_x^\infty \psi_2^T(y,\lambda)\, f(y)dy$$

$$= \psi_{11}(x,\lambda)\int_0^x u_1(y,\lambda)\, f_1(y)dy + u_1(x,\lambda)\int_x^\infty \psi_{11} f(y,\lambda) f_1(y)\,dy +$$

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$$= \psi_{21}(x,\lambda)\int_0^x u_2(y,\lambda)\, f_1(y)dy + u_2(x,\lambda)\int_x^\infty \psi_{21} f(y,\lambda) f_1(y)\, dy +$$

$$= \psi_{11}(x,\lambda)\int_0^x v_1(y,\lambda)\, f_2(y)dy + \psi_{21}(x,\lambda)\int_0^x v_2(y,\lambda)\, f_2(y)dy +$$

$$+ u_1(x,\lambda)\int_x^\infty \psi_{12}(y,\lambda)\, f_2(y)dy + u_2(x,\lambda)\int_x^\infty \psi_{22}(y,\lambda)\, f_2(y)dy$$

$$= A + B + C + D + E + F \text{ (say)} \qquad (14)$$

Where

$$A = \psi_{11}(x,\lambda)\int_0^x u_1(y,\lambda)\, f_1(y)dy + u_1(x,\lambda)\int_x^\infty \psi_{11}(y,\lambda)\, f_1(y)dy$$

$$B = \psi_{21}(x,\lambda)\int_0^x u_2(y,\lambda)\, f_1(y)dy + u_2(x,\lambda)\int_x^\infty \psi_{21}(y,\lambda)\, f_1(y)dy$$

$$C = \psi_{11}(x,\lambda)\int_0^x v_1(y,\lambda)\, f_2(y)dy$$

$$D = \psi_{21}(x,\lambda)\int_0^x v_2(y,\lambda)\, f_2(y)dy$$

$$E = u_1(x,\lambda)\int_x^\infty \psi_{12}(y,\lambda)\, f_2(y)dy$$

$$F = u_2(x,\lambda)\int_x^\infty \psi_{22}(y,\lambda)\, f_2(y)dy$$

Now

$$A = \psi_{11}(x,\lambda)\int_0^x u_1(y,\lambda)\, f_1(y)dy + u_1(x,\lambda)\int_x^\infty \psi_{11}(y,\lambda)\, f_1(y)dy$$

$$= \psi_{11}(x,\lambda)\left[\int_0^{x-\delta} + \int_{x-\delta}^{x} .\right] + u_1(x,\lambda)\left[\int_x^{x+\delta} + \int_{x+\delta}^{\infty} .\right]$$

$$= A_1 + A_2 + A_3 + A_4 \text{ (say).}$$

For |u| > J, we have from (7) and (11)

$$|\psi_{11}(y,\lambda)| \le \frac{|M_{22}(\lambda)||e^{i\mu y}|}{2|\mu|\{|M_{11}(\lambda)M_{22}(\lambda) - M_{12}(\lambda)M_{21}(\lambda)|\}}$$

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$$< \frac{\propto e^{-ty}}{|\mu|} \qquad (15)$$

where $\propto$ is a constant.

$$u_j(y,\lambda), v_j(x,\lambda) = 0(e^{tx}), (j = 1,2) \qquad (16)$$

Therefore, using (15) and (16), we have

$$A_4 = 0\left\{\frac{e^{tx}}{|\mu|} \int_{x+\delta}^{\infty} e^{-ty}|f_1(y)|\, dy\right\}$$

$$= 0\left\{\frac{e^{tx}}{|\mu|}\right\} \qquad (17)$$

The integral of (17) round the semicircle tends to zero as R $\rightarrow \infty$ for any fixed δ. Similar arguments hold for A_1 also. Now we consider A_3. For fixed x or in a finite interval, from (18).

$$X_j(x) = e^{|\mu x|} + \frac{1}{2i\mu}\int_0^x e^{1\mu(x-y)}\{p(y)X_j(y) - r(y)Y_j(y)\}\, dy +$$

$$+ \frac{1}{2i\mu}\int_x^{\infty} e^{1\mu(y-x)}\{p(y)X_j(y) - r(y)Y_j(y)\}\, dy,$$

$$Y_j(x) = e^{|\mu x|} + \frac{1}{2i\mu}\int_0^x e^{1\mu(x-y)}\{q(y)Y_j(y) - r(y)X_j(y)\}\, dy +$$

$$+ \frac{1}{2i\mu}\int_x^{\infty} e^{1\mu(y-x)}\{q(y)X_j(y) - r(y)X_j(y)\}\, dy, \qquad (18)$$

We get

$$\left|X_j(x,\lambda) - e^{i\mu x}\right| = \frac{1}{2i\mu}\left|\int_0^x e^{i\mu(x-y)}\{p(y)X_j(y,\lambda) - r(y)Y_j(y,\lambda)\}\, dy + \right.$$

$$\left. + \frac{1}{2i\mu}\int_x^{\infty} e^{i\mu(y-x)}\{p(y)X_j(y,\lambda) - r(y)Y_j(y,\lambda)\}\, dy\right|$$

$$\leq \frac{e^{-tx}}{2|\mu|}(J + \ldots\ldots) < \frac{\beta_e{}^{-tx}}{|\mu|} \text{ say, } (j=1,2) \qquad (19)$$

Similarly

$$\left|Y_j(x,\lambda) - e^{-i\mu x}\right| < \frac{\beta_e{}^{-tx}}{|\mu|}, \quad (j = 1, 2) \qquad (20)$$

$$C_{j1}(\lambda) = 1 + \frac{1}{21\mu}\int_0^{\infty} e^{-1\mu y}\{p(y)K_j(y) - r(y)Y_j(y)\}\, dy \qquad (21)$$

$$C_{j2}(\lambda) = 1 + \frac{1}{2i\mu}\int_0^{\infty} e^{-i\mu y}\{q(y)Y_j(y) - r(y)X_j(y)\}\, dy \qquad (22)$$

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Also from (21) and (22), we have

$$C_{j1}(\lambda) = \left\{1 + 0\left(\frac{1}{|\mu|}\right)\right\}, \quad (j = 1,2) \tag{23}$$

$$C_{j2}(\lambda) = \left\{1 + 0\left(\frac{1}{|\mu|}\right)\right\}, \quad (j = 1,2) \tag{24}$$

Similarly

$$\left.\begin{aligned} M_{j1}(\lambda) &= \left\{\frac{u_j(0)}{2} + 0\left(\frac{1}{|\mu|}\right),\right\} \\ M_{j2}(\lambda) &= \left\{\frac{v_j(0)}{2} + 0\left(\frac{1}{|\mu|}\right),\right\} \end{aligned}\right\} \tag{25}$$

Therefore, by using (13) and (14), (4) and (5) can be written respectively as

$$\left.\begin{aligned} X_j(x,\lambda) &= e^{i\mu x}\left\{1 + 0\left(\frac{1}{|\mu|}\right)\right\} \\ Y_j(x,\lambda) &= e^{i\mu x}\left\{1 + 0\left(\frac{1}{|\mu|}\right)\right\} \end{aligned}\right\} \tag{26}$$

Now using (25) and (26), (7) and (11) give

$$\psi_{11}(x,\lambda) = \frac{v_2(0)e^{i\mu x}\left\{1+0\left(\frac{1}{|\mu|}\right)\right\}}{i\mu[v_1(0)u_2(0) - u_1(0)v_2(0)]} \tag{27}$$

$$v_j(x,\lambda) = v_j(0)\cos\mu x + 0\left\{\frac{e^{|tx|}}{|\mu|}\right\}. \tag{28}$$

Thus from the first result of (27) and (28), we get

$$A_3 = \frac{u_1(0)v_2(0)\cos\mu x}{i\mu[v_1(0)u_2(0) - u_1(0)v_2(0)]}\int_x^{x+\delta} e^{i\mu x} f_1(y)dy + 0\left\{\frac{e^{|tx|}}{\frac{1}{|\mu|}^2}\int_x^{x+\delta} e^{-tv} f_1(y)dy\right\}$$

The last term of A_3 is

$$0\left\{\frac{1}{|\lambda|}\int_x^{x+\delta} f_1(y)dy\right\}$$

and the integral of this round the semicircle is

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$$0\left\{\int_{x}^{x+\delta} f_1(y)dy\right\}$$

which can be made as small as we please by properly choosing δ term in A3 can be written as

$$\frac{v_2(0)u_1(0)\,(e^{i\mu x}+e^{i\mu x})}{2i\mu[v_1(0)u_2(0)-u_1(0)v_2(0)]}\int_{x}^{x+\delta} e^{i\mu x}\, f_1(y)dy$$

The term involving e^{iux} also gives a zero limit by similar arguments. The remaining term is the same as in the case of an ordinary Fourier series. Similar arguments also hold for $A_{2.}$ Hence we conclude that in the bounded variation case

$$\lim_{R\to\infty}\int_{-R+i\epsilon}^{R+i\epsilon} A\, d\lambda = \frac{1}{2}\pi i\,\frac{u_1(0)v_2(0)\{f_1(x+o)+f_2(x+o)\}}{[v_2(0)u_1(0)-u_1(0)v_2(0)]}$$

Similarly

$$\lim_{R\to\infty}\int_{-R+i\epsilon}^{R+i\epsilon} B\, d\lambda = \frac{1}{2}\pi i\,\frac{v_1(0)u_2(0)\{f_1(x+o)+f_1(x+o)\}}{[v_1(0)u_2(0)-u_1(0)v_2(0)]}$$

$$\lim_{R\to\infty}\int_{-R+i\epsilon}^{R+i\epsilon} C\, d\lambda = \frac{1}{2}\pi i\,\frac{v_2(0)v_2(0)\{f_1(x+o)+f_2(x+o)\}}{[v_2(0)u_1(0)-u_1(0)v_2(0)]}$$

$$\lim_{R\to\infty}\int_{-R+i\epsilon}^{R+i\epsilon} D\, d\lambda = \frac{1}{2}\pi i\,\frac{v_1(0)v_2(0)f_2(x-0)}{[v_1(0)u_2(0)-u_1(0)v_2(0)]}$$

$$\lim_{R\to\infty}\int_{-R+i\epsilon}^{R+i\epsilon} E\, d\lambda = \frac{1}{2}\pi i\,\frac{u_1(0)u_2(0)f_2(x+0)}{[v_1(0)u_2(0)-u_1(0)v_2(0)]}$$

Thus we have

$$\lim_{R\to\infty}\int_{-R+i\epsilon}^{R+i\epsilon} \Phi_1(x,\lambda)\, d\lambda = -\frac{1}{2}\pi i\,\{f_1(x+0)+f_1(x-0)\}$$

If f (x) is continuous, then

$$f_1(x) = -\frac{1}{\pi i}\lim_{R\to\infty}\int_{-R+i\epsilon}^{R+i\epsilon}\Phi_1(x,\lambda)\, d\lambda \qquad (29)$$

Similarly

$$f_2(x) = -\frac{1}{\pi i}\lim_{R\to\infty}\int_{-R+i\epsilon}^{R+i\epsilon}\Phi_2(x,\lambda)\, d\lambda \qquad (30)$$

The above regulation are true uniformly for $0 < \epsilon \leq 1$

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EXPANSIONS

We investigate the behavior of the integrals (29) and (30) as ∈→0. Here we discuses (29) and the same arguments will apply to (30). First of all we show that (29) can be replaced by

$$f_1(x) = -\lim_{R\to\infty}\left[\frac{1}{\pi}\int_{-R-i\epsilon}^{R+i\epsilon} \text{im}\ \Phi_1(x,\lambda)d\lambda\right] \quad (31)$$

Since ϕ_1(x, λ) is analytic in the upper and lower half planes, it follows the convergence theorem that

$$f_1(x) = -\frac{1}{\pi i}\lim_{R\to\infty}\int_{-R-i\epsilon}^{R+i\epsilon} \Phi_1(x,\lambda)d\lambda \quad (32)$$

Let λ = s-i ∈ = $\bar{\lambda}$-2 i ∈, ∈ being fixed. Then from (32), we have

$$f_1(x) = -\frac{1}{\pi i}\lim_{R\to\infty}\int_{-R+i\epsilon}^{R+i\epsilon} \Phi_1(x,\bar{\lambda}-2i\epsilon)d\bar{\lambda}$$

$$= \frac{1}{\pi i}\lim_{R\to\infty}\int_{-R+i\epsilon}^{R+i\epsilon} \Phi_1(x,\bar{\lambda})d\bar{\lambda} \quad (33)$$

Adding (29) and (33) we have

$$2f_1(x) = -\frac{1}{\pi i}\lim_{R\to\infty}\int_{-R+i\epsilon}^{R+i\epsilon} \Phi_1(x,\lambda) - \Phi_1(x,\bar{\lambda})\ d\bar{\lambda}$$

$$= \frac{2}{\pi i}\lim_{R\to\infty}\int_{-R+i\epsilon}^{R+i\epsilon} [\text{im}\ \Phi_1(x,\lambda)]d\lambda$$

Because im ϕ (x,$\bar{\lambda}$) = ϕ (x, λ). The proves (31)

We know that ϕ_j(x, λ), ϕ_j (x, λ) (j = 1,2) are analytic function of λ and are real λ, it follows that each of

$$\text{im}(u_j), \text{im}(v_j), \text{im}(x_j), \text{im}(y_j) = (\epsilon) \quad (34)$$

As ϵ →0, Therefore, for x, y in the fixed interval

$$\text{im}\ \{\psi_{r1}(x,\lambda)u_j(x,\lambda)\ \psi_{r1}(y,\lambda)\} = 0(\epsilon), (1 \le r, j \le 2) \quad (35)$$

And

$$im\{\psi_{11}(x,\lambda)v_1(y,\lambda) - u_1(x,\lambda) - \psi_{12}(y,\lambda) + \psi_{21}(x,\lambda)v_2(y,\lambda) - u_2(x,\lambda) - \psi_{22}(y,\lambda)\} = 0 \quad (36)$$

Now

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$$\mathrm{im}\left[-\frac{1}{\pi}\int_{-R+i\epsilon}^{R+i\epsilon}\Phi_1(x,\lambda)\,d\lambda\right]$$

May be put in the from

$$\mathrm{im}\Bigg[-\frac{1}{\pi}\int_{-R+i\epsilon}^{R+i\epsilon}\Bigg\{\int_0^\infty\big(u_1(x,\lambda)\psi_{11}(y,\lambda)+u_2(x,\lambda)\psi_{21}(y,\lambda)\big)f_1(y)dy+$$
$$+\int_0^\infty\big(u_1(x,\lambda)\psi_{12}(y,\lambda)+u_2(x,\lambda)\psi_{22}(y,\lambda)\big)f_2(y)\,dy+$$
$$+\int_0^x\big(u_1(y,\lambda)\,\psi_{11}(x,\lambda)-u_1(x,\lambda)\psi_{11}(y,\lambda)\big)f_1(y)\,dy+$$
$$+\int_0^x\big(u_2(y,\lambda)\,\psi_{21}(x,\lambda)-u_2(x,\lambda)\psi_{21}(y,\lambda)\big)f_1(y)\,dy+$$
$$+\int_0^x\big(\psi_{11}(x,\lambda)\,v_1(y,\lambda)-u_1(x,\lambda)\,\psi_{12}(y,\lambda)\big)$$
$$+\psi_{11}(x,\lambda)\,v_2(y,\lambda)-u_2(x,\lambda)\,\psi_{22}(y,\lambda)f_2(y)\,dy\Bigg\}d\lambda\Bigg] \qquad (37)$$

Using (34), (36) and (36), (37) becomes, for fixed R and $\epsilon\to 0$

$$\mathrm{im}\Bigg[-\frac{1}{\pi}\int_{-R+i\epsilon}^{R+i\epsilon}\Bigg\{\int_0^\infty\big(u_1(x,\lambda)\psi_{11}(y,\lambda)+u_2(x,\lambda)\psi_{21}(y,\lambda)\big)f_1(y)dy+$$
$$+\int_0^\infty\big(u_1(x,\lambda)\psi_{12}(y,\lambda)+u_2(x,\lambda)\psi_{22}(y,\lambda)\big)f_2(y)\,dy\Bigg\}d\lambda\Bigg]+$$
$$+\,0\,(\epsilon) \qquad (38)$$

Now

$$\int_{-R+i\epsilon}^{R+i\epsilon}\mathrm{im}\;u_1(x,\lambda)\,d\lambda\int_0^\infty \mathrm{re}\,\psi_{22}(y,\lambda)f_1(y)dy$$

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$$- 0\,(\epsilon)\left\{\int_{R}^{R} da \int_{R}^{R} |\psi_{11}(y,s \quad i\,\epsilon) f_1(y)| dy\right\}$$

$$- 0\,(\epsilon)\left\{\int_{R}^{R}\left(\int_{0}^{\infty} |\psi_{11}(y,s - i\,\epsilon) f_1(y)| dy\right) da\right\}^{\frac{1}{2}}$$

$= 0\ \left(\epsilon^{\frac{1}{2}}\right)$ using schwarz's inequality

Similarly

$$\int_{-R}^{R} re\{u_1(x,s - i\,\epsilon)u_1(x,s)\} ds \int_{0}^{\infty} im\,\psi_{11}(y,\lambda) f_1(y) dy = 0\ \left(\epsilon^{\frac{1}{2}}\right) \tag{38}$$

Therefore (38) becomes

$$\frac{1}{\pi}\Bigg[\int_{-R}^{R} u_1(x,s)ds \int_{0}^{\infty} im\psi_{11}(y,s - i\,\epsilon) f_1(y) dy +$$
$$+ \int_{-R}^{R} u_2(x,s)ds \int_{0}^{\infty} im\psi_{21}(y,s - i\,\epsilon) f_1(y)\, dy +$$
$$+ \int_{-R}^{R} u_2(x,s)ds \int_{0}^{\infty} im\psi_{12}(y,s - i\,\epsilon) f_1(y)\, dy +$$
$$+ \int_{-R}^{R} u_2(x,s)ds \int_{0}^{\infty} im\psi_{22}(y,s - i\,\epsilon) f_2(y)\, dy\Bigg] + 0\ \left(\epsilon^{\frac{1}{2}}\right)$$
$$- \frac{1}{\pi}\Bigg[\int_{-R}^{R} u_1(x,s)ds \int_{0}^{\infty} im\psi_1^T(y,s - i\,\epsilon)\, f(y)\, dy +$$
$$+ \int_{-R}^{R} u_2(x,s)ds \int_{0}^{\infty} im\psi_2^T(y,s - i\,\epsilon)\, f(y)\, dy\Bigg]$$
$$+0\ \left(\epsilon^{\frac{1}{2}}\right) \tag{39}$$

Now let

$$\chi_r(y,\lambda) = \sum_{l=1}^{2} \int_0^{\lambda} \phi_\ell(y,s)\ dk_{r\ell}(s), \tag{40}$$

And

$$g_r(\lambda) = \int_0^{\infty} \chi_r^T(y,\lambda)\ f(y)\,dy, \tag{41}$$

The integrals in (41) exist because f (x) is L^2 [0, ∞] and Xr (y, λ) is L^2 [0, ∞]

$$\int_0^{\lambda} da \int_0^{\infty} \operatorname{im} \psi_r^T(y, s - i\,\epsilon) f(y)\,dy$$

$$= \int_0^{\infty} f^T(y) f(y)\,dy \int_0^{\lambda} \operatorname{im} \psi_r(y, s - i\,\epsilon)\ \ da$$

$\rightarrow -g_r$, as ∈→0 uniformly over a finite real λ range. Hence integrating the first term of (39) by parts, we have

$$-\frac{1}{\pi}\left[u_1(x,s) \int_0^{s} ds' \int_0^{\infty} \operatorname{im}\psi_1^T(y, s' - i\,\epsilon)\, f(y)\,dy\right]_{-R}^{R} +$$

$$+\frac{1}{\pi}\left[\int_{-R}^{R} \frac{\partial u_1(x,s)}{\partial S} ds' \int_0^{\infty} \operatorname{im}\psi_1^T(y, s' \quad i\,\epsilon)\, f(y)\,dy\right]$$

$$\rightarrow \frac{1}{\pi}[u_1(x,s)\ g_1(s)]_{-R}^{R}$$

$$-\frac{1}{\pi}\int_{-R}^{R} \frac{\partial u_1(x,s)}{\partial S} g_1(s)\ ds, \tag{42}$$

As ∈→0. If g_1 (s) (r=1,2) are of bounded variation, is equal to the Stieltjes (42) integral

$$\frac{1}{\pi}\int_{-R}^{R} u_1(x,s)\ dg_1(s)$$

Therefore, (39) becomes

$$\frac{1}{\pi}\int_{-R}^{R} u_1(x,s)\ dg_1(s) + \int_{-R}^{R} u_2(x,s)\ dg_1(s) \tag{43}$$

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In those cases which involve continuous spectrum, it has been shown by bhagat [5] that

$$\lim_{t \to 0} \operatorname{im}[m_{jj}] = \frac{\sum_{\ell=1}^{2}\{c_{k\ell}^2 + c_{k\ell}^2\}}{\lambda^{\frac{1}{2}}(A^2(\lambda) + B^2(\lambda))} \tag{44}$$

(when j=1, k=2 and when j=2, k=1)

$$\lim_{t \to 0} \operatorname{im}[m_{12}] = \lim_{t \to 0} \operatorname{im}[m_{21}] = -\frac{c_{11}c_{21}+c_{12}c_{22}+d_{11}d_{21}+d_{12}d_{22}}{\lambda^{\frac{1}{2}}(A^2(\lambda)+B^2(\lambda))} \tag{45}$$

Where

$$\left.\begin{aligned}
A(\lambda) &= c_{21}c_{12} - d_{21}d_{12} - c_{11}c_{22} + d_{11}d_{22}, \\
B(\lambda) &= c_{21}d_{12} + c_{12}d_{21} - c_{11}d_{22} - c_{21}d_{11}, \\
c_{j1}(\lambda) &= u_j(0) - \frac{1}{\mu}\int_0^{\infty}\{p(y)u_j(y) - r(y)v_j(y)\} \sin\mu y \; dy, \\
d_{j1}(\lambda) &- \frac{u_j(0)}{\mu} + \frac{1}{\mu}\int_0^{\infty}\{p(y)u_j(y) - r(y)v_j(y)\} \cos\mu y \; dy, \\
c_{j2}(\lambda) &= v_j(0) - \frac{1}{\mu}\int_0^{\infty}\{q(y)v_j(y) - r(y)u_j(y)\} \sin\mu y \; dy, \\
d_{j2}(\lambda) &= \frac{v_j'(0)}{\mu} + \frac{1}{\mu}\int_0^{\infty}\{q(y)v_j(y) - r(y)u_j(y)\} \cos\mu y \; dy,
\end{aligned}\right\} \tag{46}$$

Now substituting the values of $dk_r(s)$ $(1 \le r, l \le 2)$in (40) we have

$$\chi_r(y,\lambda) = \begin{pmatrix} \chi_{r_1}(y,\lambda) \\ \chi_{r_2}(y,\lambda) \end{pmatrix}$$

$$\int_0^{\lambda} \phi_r(y,s) \frac{\sum_{\ell=1}^{2}\{c_{k\ell}^2(s) + d_{k\ell}^2(s)\}}{s^{\frac{1}{2}}(A^2(s) + B^2(s))} ds - \int_0^{\lambda} \phi_k(y,s) \frac{c_{11}(s)c_{21}(s) + c_{12}(s)c_{22}(s) + d_{11}(s)d_{21}(s) + d_{12}(s)d_{22}(s)}{s^{\frac{1}{2}}(A^2(s) + B^2(s))} ds \tag{47}$$

Using (41) and (47), (43) becomes

$$\frac{1}{\pi}\sum_{r=1}^{2} \chi_r(y,\lambda) \int_0^{\infty} \phi_r^T(y,\lambda) \, f(y) \; dy$$

Where the integration over λ is over the interval of continuous spectrum. In the interval $-R < \lambda < 0$ $g_r(\lambda)$ (r=1,2) are constant except for a finite number of discontinuities at the poles of $m_{rs}(\lambda)(1 \le r, s \le 2)$ Hence the associated expansion is a series.

CONCLUSIONS

This paper contains a convergence theorem and obtained expansion for the vector function of the type $f(x) = \begin{bmatrix} f_1(x) \\ f_2(x) \end{bmatrix}$ which is continuous in some suitable interval and bounded variation in that interval, when *p*(x) and *q*(x) tends to +∞ or -∞, which will be suitable in the work.

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